Int. J. Solids Structures Vol. 22, No. 1, pp. 13–22, 1986
Printed in Great Britain.

0020–7683/86 $3.00+.00

IMPERFECTION SENSITIVITY OF AXIALLY COMPRESSED LAMINATED FLAT PLATES DUE TO BENDING–STRETCHING COUPLING†

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(*Received* 30 *September* 1983; *replacement received* 22 *April* 1985)

Abstract—This paper investigates the effects of the bending–stretching coupling on the imperfection sensitivity of axially compressed laminated, thin, rectangular flat plates. In particular, it is found that under certain circumstances this coupling phenomenom results in a nonzero cubic term of the potential energy, so that the structure may be imperfection sensitive depending on the sign of the imperfection. The analysis considers a nonlinear prebuckling state due to bending–stretching coupling of the structure. The buckling and initial postbuckling problem is solved using Koiter's theory of elastic stability.

1. INTRODUCTION

It is now well established that coupling between bending and stretching of certain types of composite plates may result in severe reduction of the classical buckling load of the structure (Reissner and Stavsky[1], Whitney and Leissa[2, 3] and Jones[4]). The post-buckling behavior of these structures has also been considered (Turvey and Wittrick[5], Harris[6–8], Chia and Prabhakara[9], Chia[10] and Prabhakara[11]). Much less work has been done on the influence of geometric imperfections on the initial postbuckling of laminated plates. Contrary to the common belief that flat laminated plates have stable postbuckling behavior (that is, insensitive to imperfection), the present paper aims to show that under certain circumstances they may be sensitive to imperfections in an asymptotic sense due to bending–stretching coupling effects.

The present work is motivated by the physical consideration that laminated plates may have a tendency to bend in a particular direction under axial compression due to bending–stretching coupling. Thus the equilibrium path is no longer independent of the amplitude of the buckling mode; raising speculation that the structure may be classified as an unstable asymmetric system, rather than a stable symmetric point of bifurcation within the context of single-mode analysis in the theory of elastic stability (Koiter[12], Budiansky and Hutchinson[13], Budiansky[14] and Seide[15]). If this is indeed true, then at least in the initial postbuckling regime, the structure will not be able to carry an applied load higher than the buckling load for one sign of imperfection, whereas the opposite is true for an imperfection of different sign. Example problems are chosen from equal-thickness antisymmetric cross-ply plates (Jones[4, 16] and Prabhakara[11]) and two-layered isotropic homogeneous plates under axial compression.

2. POTENTIAL ENERGY AND THE GOVERNING EQUATIONS

In order to study the initial postbuckling behavior of a laminated plate, the present analysis uses a Koiter style of approach[12] based on the principle of stationary potential energy. The potential energy of the structure under in-plane axial compression can be

† Presented at Composites: Materials and Engineering, an International Symposium, Center for Composite Materials, University of Delaware, 24–28 September 1984.

expressed in the form

$$\text{P.E.} = \frac{1}{2}\int_0^d\int_0^L \{(\overline{N}_x)[U_{,X} + \tfrac{1}{2}(W_{,X}^2)] + (\overline{N}_y)[V_{,Y} + \tfrac{1}{2}(W_{,Y}^2)] + (\overline{N}_{xy})(U_{,Y} + V_{,X} + W_{,X}W_{,Y}) - (\overline{M}_x W_{,XX}) - (\overline{M}_y W_{,YY}) - (\overline{M}_{xy} W_{,XY})\}\, dX\, dY - (t)(\overline{\sigma})\int_0^d\int_0^L U_{,X}\, dX\, dY. \quad (1)$$

where W is the out-of-plane displacement, U and V are the in-plane displacements, X and Y are the in-plane coordinates, t is the total thickness of the laminated plate, L and d are the length and width of the plate and $\overline{\sigma}$ is the applied axial stress (unit force per square length, positive for compression). By introducing the following nondimensional scheme (E_2 is the smaller of the two Young's moduli in the principle directions of an arbitrary layer, or it may be taken as an average Young's modulus),

$$\begin{gathered}(U, V) = (t^2/d)(u, v), \quad W = (t)(w), \quad (X, Y) = (d)(x, y),\\ A_{ij} = (E_2 t)(a_{ij}), \quad B_{ij} = (E_2 t^2)(b_{ij}), \quad D_{ij} = (E_2 t^3)(d_{ij}),\\ (\overline{N}_x, \overline{N}_y, \overline{N}_{xy}) = (E_2 t^3/d^2)(N_x, N_y, N_{xy}),\\ (\overline{M}_x, \overline{M}_y, \overline{M}_{xy}) = (E_2 t^4/d^2)(M_x, M_y, M_{xy}),\end{gathered} \quad (2)$$

the membrane- and bending-stress resultants are related to the displacements in the form

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy}\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16}\\ a_{12} & a_{22} & a_{26} & b_{12} & b_{22} & b_{26}\\ a_{16} & a_{26} & a_{66} & b_{16} & b_{26} & b_{66}\\ b_{11} & b_{12} & b_{16} & d_{11} & d_{12} & d_{16}\\ b_{12} & b_{22} & b_{26} & d_{12} & d_{22} & d_{26}\\ b_{16} & b_{26} & b_{66} & d_{16} & d_{26} & d_{66}\end{bmatrix} \begin{bmatrix} u_{,x} + \frac{1}{2}(w_{,x}^2) \\ v_{,y} + \frac{1}{2}(w_{,y}^2) \\ (u_{,y} + v_{,x}) + (w_{,x}w_{,y}) \\ -w_{,xx} \\ -w_{,yy} \\ (-2)(w_{,xy})\end{bmatrix}. \quad (3)$$

The above equation can easily be expressed in the partially inverted nondimensional form

$$\begin{bmatrix} u_{,x} + \frac{1}{2}(w_{,x}^2) \\ v_{,y} + \frac{1}{2}(w_{,y}^2) \\ (u_{,y} + v_{,x}) + w_{,x}w_{,y} \\ M_x \\ M_y \\ M_{xy}\end{bmatrix} = \begin{bmatrix} a_{11}^* & a_{12}^* & a_{16}^* & b_{11}^* & b_{12}^* & b_{16}^*\\ a_{12}^* & a_{22}^* & a_{26}^* & b_{21}^* & b_{22}^* & b_{26}^*\\ a_{16}^* & a_{26}^* & a_{66}^* & b_{61}^* & b_{62}^* & b_{66}^*\\ -b_{11}^* & -b_{21}^* & -b_{61}^* & d_{11}^* & d_{12}^* & d_{16}^*\\ -b_{12}^* & -b_{22}^* & -b_{62}^* & d_{12}^* & d_{22}^* & d_{26}^*\\ -b_{16}^* & -b_{26}^* & -b_{66}^* & d_{16}^* & d_{26}^* & d_{66}^*\end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ -w_{,xx} \\ -w_{,yy} \\ (-2)(w_{,xy})\end{bmatrix}, \quad (4)$$

where the nondimensional coefficients a_{ij}^*, b_{ij}^* and d_{ij}^* are related to a_{ij}, b_{ij} and d_{ij} by

$$\begin{aligned}(a_{ij}^*) &= (a_{ij})^{-1} = (E_2 t)(A_{ij}^*),\\ (b_{ij}^*) &= -(a_{ij})^{-1}(b_{ij}) = (1/t)(B_{ij}^*),\\ (d_{ij}^*) &= (d_{ij}) - (b_{ij})(a_{ij})^{-1}(b_{ij}) = [1/(E_2 t^3)](D_{ij}^*)\end{aligned} \quad (5)$$

It should be noted that $a_{ij}^* = a_{ji}^*$, $d_{ij}^* = d_{ji}^*$ but in general $b_{ij}^* \neq b_{ji}^*$.

Introducing the Airy stress function F, defined to be $[F = (E_2 t^3)(f)]$,

$$N_x = f_{,yy}, \quad N_y = f_{,xx}, \quad N_{xy} = -f_{,xy}, \quad (6)$$

the governing equilibrium and compatibility equations are, respectively (Stavsky and Hoff ([17], p. 21),

$$
\begin{aligned}
L_{d^*}(w) + L_{b^*}(f) &= f_{,yy}w_{,xx} + f_{,xx}w_{,yy} - (2)(f_{,xy})(w_{,xy}), \\
L_{a^*}(f) - L_{b^*}(w) &= (w_{,xy})^2 - (w_{,xx})(w_{,yy}).
\end{aligned} \tag{7}
$$

Furthermore, the nondimensional linear differential operators L_a, L_b and L_d are defined as

$$
\begin{aligned}
L_{a^*}(\ ) &= a_{22}^*(\ )_{,xxxx} - (2a_{26}^*)(\ )_{,xxxy} + (2a_{12}^* + a_{66}^*)(\ )_{,xxyy} \\
&\quad - (2a_{16}^*)(\ )_{,xyyy} + a_{11}^*(\ )_{,yyyy}, \\
L_{b^*}(\ ) &= b_{21}^*(\ )_{,xxxx} + (2b_{26}^* - b_{61}^*)(\ )_{,xxxy} + (b_{11}^* + b_{22}^* - 2b_{66}^*)(\ )_{,xxyy} \\
&\quad + (2b_{16}^* - b_{62}^*)(\ )_{,xyyy} + b_{12}^*(\ )_{,yyyy}, \\
L_{d^*}(\ ) &= d_{11}^*(\ )_{,xxxx} + (4d_{16}^*)(\ )_{,xyyy} + (2d_{12}^* + 4d_{66}^*)(\ )_{,xxyy} \\
&\quad + (4d_{26}^*)(\ )_{,xyyy} + d_{22}^*(\ )_{,yyyy}.
\end{aligned} \tag{8}
$$

At this stage it should be pointed out that the laminated plate under axial compression will bend prior to buckling, due to bending–stretching coupling via the b_{ij}^* terms. As the aim of the present analysis is to investigate the influence of the above coupling on the imperfection sensitivity of the structure in a relatively straight-forward manner, consideration is restricted to the simplified case when the aspect ratio L/d is sufficiently large so that the boundary conditions at $X = 0,L$ may be neglected. On the remaining edges $y = 0,1$, the boundary conditions are

$$
\begin{aligned}
&w(y = 0) = w(y = 1) = 0, \qquad M_y(y = 0) = M_y(y = 1) = 0, \\
&N_y(y = 0) = N_y(y = 1) = 0, \qquad \epsilon_x(y = 0) = \epsilon_x(y = 1) = \text{const},
\end{aligned} \tag{9}
$$

where ϵ_x is the axial strain.

Thus, from eqn (4), the condition $M_y(y = 0) = M_y(y = 1) = 0$ implies that the prebuckling displacement w_p is [6],

$$
w_p = (c_0)(y^2) - c_0 y \qquad \text{where } c_0 = (\sigma/2)(b_{12}^*/d_{22}^*), \tag{10}
$$

so that w_p (at $y = 1/2$) $= -c_0/4$. Furthermore, the prebuckling stress function f_p,

$$
f_{p,yy} = -\sigma, \qquad f_{p,xx} = 0 \qquad f_{p,xy} = 0, \tag{11}
$$

is valid everywhere on the plate. It may be observed that the above prebuckling state satisfies the governing differential equations. It should be mentioned that the existence of prebuckling deformation, which satisfies the governing differential equations, will not indicate whether the prebuckling state is linear or nonlinear. Rather, a nonlinear prebuckling state is associated with one that involves bending of the plate prior to buckling. Thus $w_{p,y} \neq 0$ is classified as a nonlinear prebuckling state, whereas a linear prebuckling state is indicated by $w_p = 0$ or $w_p = \text{const}$.

Finally, the above prebuckling deformation is also valid for moderate values of aspect ratio provided w_p is unrestrained along the loading edges (defined by $x = 0$ and $x = L/d$) except that $w_p = 0$ at $y = 0$ and at $y = 1$.

3. CLASSICAL BUCKLING LOAD

The classical buckling load is obtained by minimizing the quadratic terms of the potential energy. In terms of the mixed formulation, it is found by solving the linearized form of the governing differentiations (substitution of $w = w_p + w_c$ and $f = f_p + f_c$, and then linearizing with respect to w_c and f_c),

$$\begin{aligned}[L_{d^*}(w_c) + (\sigma)(w_{c,xx})] - [-L_{b^*}(f_c) + (2c_0)(f_{c,xx})] &= 0, \\ [-L_{b^*}(w_c) + (2c_0)(w_{c,xx})] + L_{a^*}(f_c) &= 0.\end{aligned} \tag{12}$$

As the main objective of the present study is to investigate the influence of the coupling coefficients b_{ij}^*, and in order to avoid a strictly numerical solution for the buckling problem, all the "16" and "26" terms are set to zero, that is,

$$a_{16}^* = a_{26}^* = b_{16}^* = b_{26}^* = d_{16}^* = d_{26}^* = 0 \tag{13}$$

By doing so, the buckling mode [which satisfies eqns (9)] can be sought in the sinusoidal form

$$\begin{bmatrix} w_c(x, y) \\ f_c(x, y) \end{bmatrix} = (\delta/t) \begin{bmatrix} \sin(M\pi x)\sin(n\pi y) \\ \beta \sin(M\pi x)\sin(n\pi y) \end{bmatrix}, \tag{14}$$

where δ is the amplitude of the buckling mode, $M \equiv (m)(d/L)$ and the integers m and n are the number of half waves in the x and y directions, respectively. Substituting the buckling mode into the linearized differential equations, the two algebraic equations in two unknowns σ and β are

$$\begin{aligned}[C_{d^*} - (M/\pi)^2(\sigma)] + [C_{b^*} + (b_{12}^*/d_{22}^*)(M/\pi)^2(\sigma)](\beta) &= 0, \\ [-C_{b^*} - (b_{12}^*/d_{22}^*)(M/\pi)^2(\sigma)] + \beta C_{a^*} &= 0,\end{aligned} \tag{15}$$

where

$$\begin{aligned} C_{a^*} &= a_{22}^* M^4 + (2a_{12}^* + a_{66}^*)(M^2 n^2) + a_{11}^* n^4, \\ C_{b^*} &= b_{21}^* M^4 + (b_{11}^* + b_{22}^* - 2b_{66}^*)(M^2 n^2) + b_{12}^* n^4, \\ C_{d^*} &= d_{11}^* M^4 + (2d_{12}^* + 4d_{66}^*)(M^2 n^2) + d_{22}^* n^4. \end{aligned} \tag{16}$$

Upon eliminating β, the classical buckling load can be obtained by solving for the smallest positive real root of the quadratic equation, for all permissible wave numbers,

$$(b_{12}^*/d_{22}^*)^2(M/\pi)^4(\sigma_c)^2 - (M/\pi)^2[C_{a^*} - (2C_{b^*})(b_{12}^*/d_{22}^*)](\sigma_c) + [C_{a^*}C_{d^*} + (C_{b^*})^2] = 0. \tag{17}$$

The corresponding ratio of the eigenvector is

$$\beta = (1/C_{a^*})[C_{b^*} + (b_{12}^*/d_{22}^*)(M/\pi)^2(\sigma)]. \tag{18}$$

4. INITIAL POSTBUCKLING BEHAVIOR

Following a Koiter style of analysis[12], incorporating the possibility of a nonlinear prebuckling state, the potential energy of the structure can be expanded in terms of a

Taylor series at the classical buckling load in the form

$$\text{P.E.}/(E_2t^3) = \{[(\sigma - \sigma_c)(e_1) + \tfrac{1}{2}(\sigma - \sigma_c)^2(e_2)](\delta/t)^2 + (P_3^0[u_c] + P_3'[u_c])(\delta/t)^3 + Q\}(t/d)^2, \quad (19)$$

where

$$e_1 = \frac{\partial}{\partial\sigma}(P_2'[u_c]) \quad \text{and} \quad e = \frac{\partial^2}{\partial\sigma^2}(P_2''[u_c]) \quad (20)$$

Furthermore, Q represents terms which involve the initial geometric imperfection (whichj is taken to be of the same form as the buckling mode) and unlike the usual Koiter notation, the applied load σ is incorporated into the $P_2'[u_c]$, $P_2''[u_c]$ and $P_3'[u_c]$ expressions. By minimizing the potential energy with respect to δ/t, the equilibrium equation becomes, (upon dividing through by $-2e_1\sigma_c$),

$$[1 - (\sigma/\sigma_c)]\{1 + [1 - (\sigma/\sigma_c)](e_3)\}(\delta/t) + (a^*)(\delta/t)^2 = (\sigma/\sigma_c)(\bar{\delta}/t),$$

where

$$e_3 = (-\sigma_c/4)(e_2/e_1), \quad (22)$$

$$a^* = (3)(P_3^0[u_c] + P_3'[u_c])/(-2e_1\sigma_c). \quad (23)$$

In the above equilibrium equation, $\bar{\delta}$ is the amplitude of the imperfection and the influence of nonlinear prebuckling state on the imperfection terms as well as higher-order terms have been neglected. Moreover, it should be noted that in the special case of linear prebuckling state,

$$P_3'[u_c] = 0 \quad \text{and} \quad e_2 = e_3 = 0, \quad (24)$$

and thus, a^* converges to the well known "a" coefficient (Budiansky and Hutchinson[13]).

Evidently, in order to obtain the a^* coefficient, it is necessary to compute e_1, $P_3^0[u_c]$ and $P_3'[u_c]$. Denoting the cubic and quartic terms of the potential energy (prior to substitution of $u = u_p + u_c$, $v = v_p + v_c$ and $w = w_p + w_c$) by V_3 and V_4, respectively, it is clear that e_1 and $P_3^0[u_c]$ can be obtained from V_3, while $P_3'[u_c]$ can be obtained from V_4. In terms of the mixed formulation, the relevant energy terms are (taking into account the prebuckling deformation w_p is independent of the axial x coordinate),

$$(P_2'[u_c])(\delta/t)^2 = \frac{1}{2}\int_0^1\int_0^{L/d}\{(-\sigma/2)(w_{c,x})^2 + [(w_{p,y})(w_{c,y})] \times [f_{c,xx} - b_{12}w_{c,xx} - b_{22}w_{c,yy} - (2b_{26})(w_{c,xy})]\}\,dx\,dy, \quad (25)$$

$$P_2''[u_c] = \frac{1}{4}\int_0^1\int_0^{L/d}\{(w_{p,y})^2[(a_{12} + 2a_{66})(w_{c,x})^2 + (3a_{22})(w_{c,y})^2 + (6a_{26})(w_{c,x})(w_{c,y})]\}\,dx\,dy, \quad (26)$$

$$(P_3^0[u_c])(\delta/t)^3 = -\frac{1}{2}\int_0^1\int_0^{L/d}\{(w_{c,x})^2(b_{11}w_{c,xx} + b_{12}w_{c,yy} + 2b_{16}w_{c,xy}) + (w_{c,y})^2(b_{12}w_{c,xx} + b_{22}w_{c,yy} + 2b_{26}w_{c,xy} + (2)(w_{c,x})(w_{c,y}) \times [b_{16}w_{c,xx} + b_{26}w_{c,yy} + 2b_{66}w_{c,xy}] + (\cdots)\}\,dx\,dy \quad (27)$$

and

$$(P_3'[u_c])(\delta/t)^3 = \frac{1}{2}\int_0^1 \int_0^{L/d} \{[(a_{12} + 2a_{66})(w_{c,x})^2(w_{c,y}) + (a_{22})(w_{c,y})^3 + (a_{16})(w_{c,x})^3 + (3a_{26})(w_{c,x})(w_{c,y})^2][w_{p,y}]\,dx\,dy, \tag{28}$$

where terms not explicitly written down in $P_3^0[u_c]$ vanish upon integration and a_{ij} and b_{ij} are not to be confused with a_{ij}^* and b_{ij}^*. Substituting the buckling mode w_c and f_c into the energy expressions, the appropriate terms which are needed to compute the a^* coefficient and the equilibrium equation are

$$\begin{aligned}
P_3^0[u_c] &= \begin{cases} (L/d)(4/9)[\pi^2/(mn)](Z_B) & \text{if } m \text{ is odd,} \\ 0 \quad \text{if } m \text{ is even,} \end{cases} \\
P_3'[u_c] &= (b_{12}^*/d_{22}^*)(-4/27)(\sigma)(Z_A), \\
e_1 &= (-1/16)(m\pi)^2(d/L)(1 + Y_B), \\
e_2 &= (\pi^2/16)(b_{12}^*/d_{22}^*)^2(L/d)(Y_A), \\
e_3 &= (-\sigma_c/4)(e_2/e_1),
\end{aligned} \tag{29}$$

where Y_A, Y_B, Z_A and Z_B are defined to be

$$\begin{aligned}
Y_A &= (a_{12} + 2a_{66})(M^2)\{(1/6) - 1/(n^2\pi^2)\} \\
&\quad + (3n^2a_{22})\{(1/6) + [1/(n^2\pi^2)]\}, \\
Y_B &= (b_{12}^*/d_{22}^*)[(b_{12} - \beta) + (b_{22})(n/M)^2], \\
Z_A &= (a_{12} + 2a_{66})(M/n) + (7a_{22})(n/M), \\
Z_B &= b_{11}M^4 + b_{22}n^4 + (2)(b_{12} - b_{66})(M^2n^2).
\end{aligned} \tag{30}$$

In the special case of a structure exhibiting linear prebuckling state, the above result shows that the existence of the a^* coefficient depends on B_{11}, B_{22}, B_{12} and B_{66} via Z_B, since $P_3'[u_c] = 0$. Thus, in the case of antisymmetric angle-ply plates characterized by

$$a_{16} = a_{26} = d_{16} = d_{26} = b_{11} = b_{12} = b_{22} = b_{66} = 0, \tag{31}$$

it is clear that a^* is identically zero. On the other hand, for an antisymmetric cross ply characterized by

$$\begin{aligned}
&a_{16} = a_{26} = d_{16} = d_{26} = b_{16} = b_{26} = b_{12} = b_{66} = 0, \\
&b_{22} = -b_{11},
\end{aligned} \tag{32}$$

it can be seen that $P_3'[u_c]$ will in general not vanish (even though $P_3^0[u_c]$ may vanish, since in many practical situations $M = n = 1$), resulting in a nonzero a^* coefficient.

In the event a^* turns out to be zero or negligibly small, the determination of the imperfection sensitivity of the structure depends on the coefficient of the quartic term. Judging from the existing literature (Harris[6–8]), it appears that this quartic term is, in general, positive. Thus assuming this general case, the system will snap or pop to another stable configuration for a nonzero a^* coefficient, whereas no such snapping will occur if $a^* = 0$.

5. EXAMPLE PROBLEMS

In order to exhibit the influence of the bending–stretching coupling on the imperfection sensitivity of the laminated plate, two example problems are presented in this section: antisymmetric equal-thickness cross-ply plates, and laminates consisting of two equal-thickness isotropic homogeneous layers.

The first type of structure being considered is antisymmetric equal-thickness cross-ply plates (Jones[4, 6] and Prabhakara[11]), where the a_{ij}, b_{ij} and d_{ij} matrices are in the form of eqn (32). The parameters held fixed are

$$G_{12}/E_2 = 0.5 \quad \text{and} \quad \nu_{12} = 0.25 \tag{33}$$

(where G_{12} is the shear modulus and ν_{12} is the Poisson ratio) while the more important parameters E_1/E_2 and the number of layers N which influence the bending–stretching behavior is permitted to vary. Figure 1(a) shows a graph of nondimensional buckling load $\sigma_c = \bar{\sigma}_c/(E_2t^2/d^2)$ vs E_1/E_2 for N being 2, 4, 6 and infinity. It can be seen that the curve for $N = 2$ is substantially below the others (especially in the region where E_1/E_2 is large), indicating the degrading influence of the b_{11} term. It should be noted that although the prebuckling deformation has practically no effect on the classical buckling load (they agree to about four significant digits), it will be shown that its role on the a^* coefficient is indispensible in the imperfection sensitivity study of the present laminated structure.

The corresponding a^* coefficient for $N = 2$, 4 and 6 ($a^* = 0$ for N = infinity) are plotted in Fig. 1(b). Although the buckling load is essentially independent of the aspect ratio L/d, the a^* coefficient may be significantly reduced for a longer plate. This is because of the fact that the optimal wave numbers m and n generally turn out to be L/d and one, respectively, so that $P_3^1[u_c]$ is independent of L/d (note that $P_3^0[u_c] = 0$ for the present optimal wave-number combinations), while e_1 increases with L/d. Judging from the magnitude of the a^* coefficient, the cross-ply plate appears to be most imperfection sensitive in the vicinity of E_1/E_2 being approximately four, characterized by the peak in the a^* curve, rather than at large values of E_1/E_2 where the coupling effect is most degrading in reducing the buckling load. This somewhat unexpected result arises from the fact that $P_3^0[u_c]$ generally turns out to be small, so that a^* depends a great deal on $P_3^1[u_c]$.

The second type of laminated structure being considered consists of two homogeneous isotropic plates of equal thickness (Ashton, Halpin and Petit[18], p. 43). The

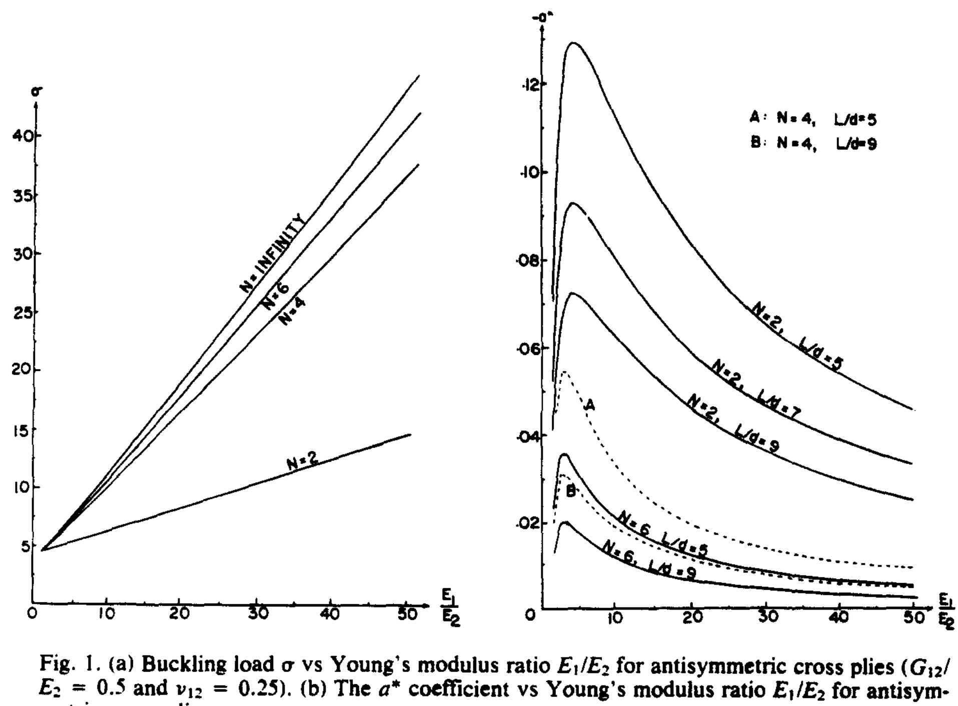

Fig. 1. (a) Buckling load σ vs Young's modulus ratio E_1/E_2 for antisymmetric cross plies ($G_{12}/E_2 = 0.5$ and $\nu_{12} = 0.25$). (b) The a^* coefficient vs Young's modulus ratio E_1/E_2 for antisymmetric cross plies.

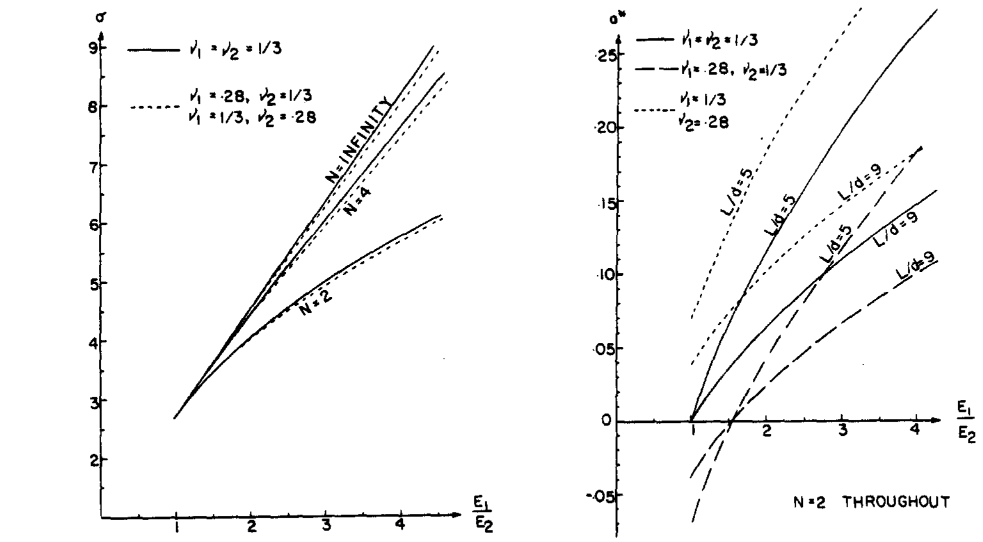

Fig. 2. (a) Buckling load σ vs Young's modulus ratio between adjacent layers in laminated homogeneous isotropic plates. (b) The a^* coefficient vs Young's modulus ratio between adjacent layers in laminated homogeneous isotropic plates.

Young's moduli for the first and second layer are E_1 and E_2, while the corresponding Poisson's ratios are denoted by ν_1 and ν_2. Keeping the total thickness t of the laminated plate fixed, the effect of the bending–stretching coupling can be shown by increasing the number of layers N (taken to be an even integer in an antisymmetric layup). The laminated plate is characterized by

$$a_{16} = a_{26} = b_{16} = b_{26} = d_{16} = d_{26} = 0, \tag{34}$$

while the boundary conditions are identical to that examined in the first example problem. In addition to the ratio of the Young's moduli E_1/E_2 being permitted to vary, three types of Poisson's ratio combinations are investigated:

$$\begin{aligned} &\nu_1 = 0.28 \quad \text{and} \quad \nu_2 = 1/3, \\ &\nu_1 = 1/3 \quad \text{and} \quad \nu_2 = 0.28, \\ &\nu_1 = \nu_2 = 1/3. \end{aligned} \tag{35}$$

Figure 2(a) shows the nondimensional buckling load defined by $\sigma = \bar{\sigma}(E_2 t^2/d^2)$ vs E_1/E_2 for various Poisson's ratio combinations. The effect of different Poisson's ratio combinations on the buckling load are found to be very small. However, for sufficiently large value of E_1/E_2, there are significant reduction of the buckling load as the number of layers is reduced from infinity, due to bending–stretching coupling. Moreover, the

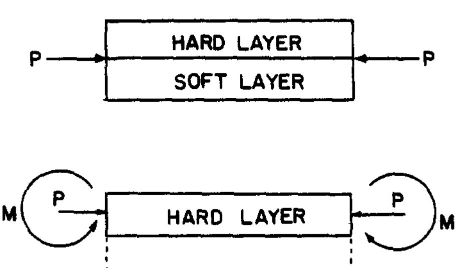

Fig. 3. Schematic diagram showing the equivalent representation of a two-layer equal-thickness laminated plate under compression.

buckling load is essentially independent of the aspect ratio L/d, and the optimum wave numbers m and n are found to be $m = L/d$ and $n = 1$. As a check on the analysis, the buckling load for equal Young's moduli and equal Poisson's ratio agrees with the well-known formula for the buckling of a homogeneous isotropic flat plate under axial compression:

$$\sigma = (E_2)(t/d)^2 \left(\frac{\pi^2}{3(1 - \nu^2)}\right) . \tag{36}$$

The corresponding a^* coefficients (e_3 can generally be considered as very small) vs E_1/E_2 for a two-ply laminated plate, with aspect ratio L/d being 5 and 9, are plotted in Fig. 2(b). It can be seen that the magnitude of a^* increases with E_1/E_2. Furthermore, it decreases with increasing aspect ratio so that in the case of infinite aspect ratio, the a^* coefficient vanishes. It is found that in addition to the reduction of the buckling load, the bending–stretching coupling effect is also responsible for a higher degree of imperfection sensitivity, as it is evident by a larger magnitude of the a^* coefficient as E_1/E_2 increases. It is interesting to observe that the prebuckling state is actually linear (no bending prior to buckling) in the case of equal Poisson's ratio for all values of E_1/E_2, so that $P_3'[u_c] = 0$. While the effect of different Poisson's ratio combinations on the buckling load is very small, their influence on the a^* coefficients may be quite pronounced. Finally from the buckling loads in Fig. 2(a) and magnitudes of the a^* coefficients for a four-layer laminated plate [not shown in Fig. 2(b)], it can be concluded that the degrading effect of bending–stretching coupling effect rapidly diminishes as the number of layers is increased.

Figure 3(a) shows a schematic diagram of the two-layer antisymmetric laminated plate which exhibits bending–stretching coupling under longitudinal compression. The axial load is applied at the center of the entire laminated plate. The hard layer may represent steel and the soft layer aluminum. For an antisymmetric cross-ply rectangular plate, the hard layer is the one where the fibers are in the longitudinal direction (same direction as the applied load), and the soft layer in the transverse direction. For purposes of illustration, the soft layer is assumed to be completely ineffective in resisting the axial load, and thus the entire loaded structure is equivalent to a longitudinal load applied at the middle of the hard layer with an additional bending moment M [see Fig. 3(b)]. Thus there is a particular direction in which the plate will tend to buckle due to this applied moment, so that the structure is no longer symmetric (that is, it will have asymmetric postbuckling behavior). Furthermore, the cubic term of the potential energy expression of the laminated plate will no longer be zero, due to the asymmetric nature of the problem. The above discussion is applicable only for nonsquare rectangular plates, since the problem remains symmetric for square laminated plates. Finally, similar behavior will occur for antisymmetric equal-thickness laminates with the number of layers being 2, 4, 6, . . . , and the asymmetric effects diminish for larger number of layers. It can be seen by the above reasoning that the cubic term of the potential energy is identically zero for an antisymmetric angle-ply rectangular plate.

The asymmetric postbuckling behavior of anti-symmetric cross-ply rectangular plates under shear loads[19] and symmetric cross-ply, short cylindrical panels under compression[20] were reported by the author in separate papers.

6. CONCLUDING REMARKS

A theoretical study for the effect of the bending–stretching coupling coefficient on the imperfection sensitivity behavior in laminated plates has been considered. Judging from the results of the example problems, the coupling terms may lead to a nonzero cubic term of the potential energy, thus reducing the load carrying capacity of the structure. It should be cautioned that the influence of modes, other than the one corresponding to the classical buckling load, is neglected. Moreover, the analysis is valid only for sufficiently small values of the imperfection amplitude, so that the region of validity in the case of small values of the magnitude of a^* may be extremely limited.

Finally, it is conjectured that the influence of torsional rigidity of the stringers (Hui and Hansen[21]) located at the edges $y = 0, 1$ may significantly raise the buckling load and reduce the magnitude of the a^* coefficient. Moreover, it should be cautioned that the degrading effect of bending–stretching coupling on the imperfection sensitivity of the structure may also be applicable to laminated cylindrical panels under axial compression (Hui[22]). Further investigations in the imperfection sensitivity of fiber reinforced cylindrical shells with a nonlinear prebuckling state due to barreling of the shell (Khot and Venkayya[23] and Tennyson *et al.*[24]) are in progress.

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